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MD88-32-IB
TR88-32
BN-1089

September 1988

INSTITUTE FOR PHYSICAL SCIENCE
AND TECHNOLOGY

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by

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[1]Research partially supported by the U.S. Office of Naval Research under Contract N-00014-85-K-0169.

[2]Research partially supported by the National Science Foundation under Grant DMS-85-16191 during the stay at the Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742.

[3]Research partially supported by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under Grant Number AFOSR 85-0322.

Abstract. Various methods for treating nonhomogeneous Dirichlet boundary conditions for the p-version of the finite element method are presented. These methods are theoretically and computationally analyzed. Numerical experimentations are given. They clearly illustrate the importance of the right treatment of the nonhomogeneous Dirichlet boundary conditions.

1. Introduction.

The classical finite element method approximates the solution of the problem by piecewise polynomials (usually of degree $p = 1,2$). Accuracy is then achieved by decreasing the elements size h, which is why this method is often called the h-version. The h-version has been thoroughly investigated both theoretically and computationally and many codes based on this approach are available.

In recent years, attention has been focused on two new finite element developments, the p-version when the mesh is fixed with accuracy being achieved by increasing the degree p of the elements either uniformly or selectively, and the h-p-version which increases the degree of elements and modifies the mesh simultaneously. The first theoretical papers addressing these new versions appeared in 1981 ([1] and [2]) and discussed basic approximation and convergence results for these methods. (See also [3], [4] for the state of art of these methods.) These methods are related to the spectral methods which are now used in computational fluid mechanics (see eg. [5]).

In structural mechanics the equations to be solved are of elliptic type and the input data are piecewise analytic. This piecewise analyticity has profound impact on the regularity of the solution. The solution is piecewise analytic with singular behavior in a priori known areas such as neighborhoods of corners, see [6]. It can be shown ([7], [8]) that for this type of problem the p-version has a rate of convergence (in the energy norm) which is twice that of the h-version (with respect to the number of the degrees of freedom) when a quasiuniform mesh is used. The h-p-version then leads to an exponential rate of convergence. For more, see [9], [10].

The p and h-p versions have been implemented in the commercial code PROBE (Noetic Technology, Inc., St. Louis) which is used in the industries.

The program STRIPE (Aeronautical Research Institute, Sweden) is also based on the p and h-p version.

In [11], [12], [13], we discussed a class of theoretical questions related to the implementation of essential boundary conditions in the framework of the p and h-p versions. The main aspect is to approximate the nonhomogeneous boundary conditions so that they will be in the trace spaces of the finite element subspaces. This approximation can be made in various ways. We introduce in this paper a one parametric family of approximations based on the expansion in Jacobi polynomials which naturally lead to the employment of the theory of weighted Sobolev spaces. We analyze this family theoretically and perform various comparative numerical studies.

Section 2 of the paper introduces the weighted spaces, and interpolation results between families of these spaces. It also contains a family of projection operators based on Jacobi polynomials which are analyzed both theoretically and computationally in section 3. These results are applied to the p-version of the finite element for solving problems with nonhomogeneous Dirichlet boundary data in two dimensions. The results of numerical experiments are also presented in this section.

2. The function spaces.

2.1. Notation.

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain with straight or curved sides. We assume that $\partial\Omega = \Gamma = \bigcup_{i=1}^{M} \bar{\Gamma}_i$ where $\bar{\Gamma}_i$, $i = 1,2,\dots, M$ is an analytic simple arc connecting the vertices A_{i-1} and A_i $(A_0 = A_M)$. By Γ_i we denote $\bar{\Gamma}_i - (A_{i-1} \cup A_i)$ and by ω_i the interior angle of A_i, $0 < \omega_i < 2\pi$. The notation scheme is shown in Figure 2.1. Further we denote $I = (-1,1) \subset \mathbb{R}^1$.

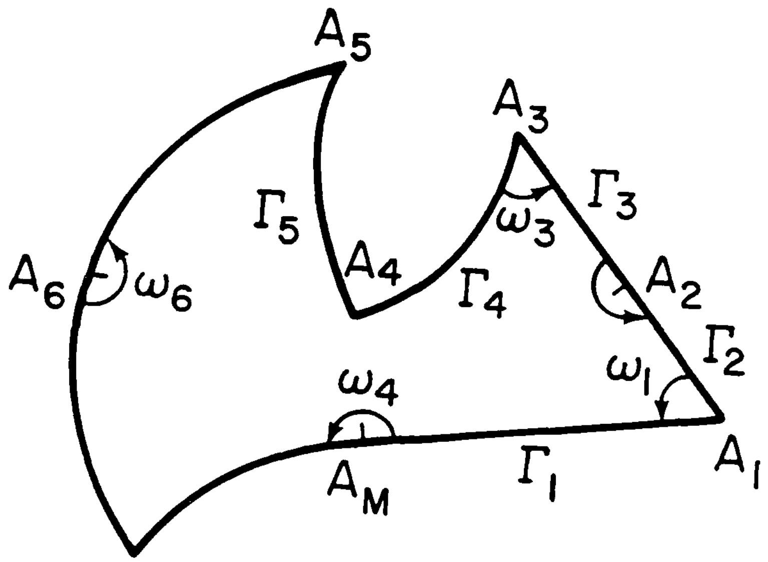

Figure 2.1. Scheme of the domain and the notation.

2.2. The spaces.

We will consider the usual Sobolev spaces $L_2(\Omega) = H^0(\Omega)$, $H^k(\Omega)$, $\mathring{H}^k(\Omega)$, $k > 0$ integer. The norm and seminorm will be denoted by $\|\cdot\|_{H^k(\Omega)}$ and $|\cdot|_{H^k(\Omega)}$ respectively. We will also consider the spaces with k fractional. The definition then is the usual one based on the K-method, see eg. [14], [15].

On I we will consider a family of weighted spaces. For $s \geq 0$ an integer and μ, ν real we define the norm $\|\cdot\|_{W^s(\mu,\nu)}$ by

$$(2.1a) \qquad \|u\|^2_{W^s(\mu,\nu)} = \int_{-1}^{+1} [(1-x^2)^{-\mu}\left(\frac{d^s u}{dx^s}\right)^2 + (1-x^2)^{-\nu} u^2]dx.$$

For $0 < s = [s] + \{s\}$ where $[s]$ is an integer and $0 < \{s\} < 1$ we define

(2.1b)

$$\|u\|^2_{W^s(\mu,\nu)} = \|u\|^2_{W^s(\mu,\nu)} + \int_{-1}^{1}\int_{-1}^{1} \frac{[(1-x^2)^{-\mu/2}u^{(s)}(x)-(1-y^2)^{-\mu/2}u^{(s)}(y)]^2}{|x-y|^{1+2\{s\}}}dxdy$$

where $u^{(s)} = \frac{d^s u}{dx^s}$. For $s = 0$ we use $\mu = \nu$ and

$$\|u\|^2_{W^0(\mu,\nu)} = \int_{-1}^{1} (1-x^2)^{-\mu}u^2 dx$$

(instead $2\int(1-x^2)^{-\mu}u^2dx$)

Let further C^∞, respectively C_0^∞, denote the set of functions with all derivatives on $\bar{I} = [-1,1]$, respectively functions with all derivatives and compact support in I.

We define now $W^s(\mu,\nu)$ and $\mathring{W}^s(\mu,\nu)$ as the completion of the set $\{u \in C^\infty \mid \|u\|_{W^s(\mu,\nu)} < \infty\}$, resp. of the set C_0^∞. The spaces $W^s(\mu,\nu)$ are Hilbert spaces. $W^s(0,0)$ is the usual Sobolev space with fractional derivatives, i.e., $W^s(0,0) = H^s(I)$. We will consider in the sequel various spaces and their equivalency. The equivalency will be denoted by $\doteq$. The weighted spaces $W^s(\mu,\nu)$ have been studied in a general context in [14]. Let us mention now some of their properties which will be used later.

Theorem 2.1.

(a) Let $\{s\} \neq 1/2$. Then all spaces $\mathring{W}^s(\mu,\nu)$ with $-\infty < \nu < 2s + \mu$

are equivalent. If in addition $\mu + 2s \neq 1 + 2k$, $k = 0, \ldots, [s] - 1$, then these spaces are equal to $W^s(\mu, \mu+2s)$.

(b) If $\nu \geq \mu + 2s$ then $W^s(\mu, \mu+2s) = \mathring{W}^s(\mu, \mu+2s)$.

(c) If $s = 0$ then $\mathring{W}^0(\mu, \nu) = W^0(\mu, \nu)$ for any $-\infty < \nu < \infty$..

For the proof, see Theorem 3.2.6 and Remark 6-3.2.6 of [14]. □

Let $(H_1, H_2)_{\theta,2}$ be the interpolated space by the K-method. For more see [14], [15]. Then it is well known that for $0 < \{s\} < 1$, $s \geq 0$,

$$H^{[s]+\{s\}} = (H^{[s]}, H^{[s]+1})_{\{s\},2}, \tag{2.2a}$$

and for $\{s\} \neq 1/2$

$$\mathring{H}^{[s]+\{s\}} = (\mathring{H}^{[s]}, \mathring{H}^{[s]+1})_{\{s\},2}. \tag{2.2b}$$

By H^s we denoted the standard Sobolev spaces $H^s(\Omega)$ or $H^s(I)$. We have then

$$H^{[s]+\{s\}}(I) = W^{[s]+\{s\}}(0,0),$$

respectively

$$\mathring{H}^{[s]+\{s\}}(I) = \mathring{W}^{[s]+\{s\}}(0,0)$$

In the case $\{s\} = 1/2$ the interpolated norm $\mathring{H}^{[s]+1/2}(I)$ has the form

$$\|u\|^2_{\mathring{H}^{[s]+1/2}} = \left[\|u\|^2_{H^{[s]+1/2}} + \|(1-x^2)^{-1/2}u^{(s)}\|_{L_2(I)}\right].$$

This norm is usually denoted by $H^{1/2}_{00}(I)$. We will consider only the interpolated norms in this paper. Hence $\mathring{W}^{[s]+1/2}(0,0)$ not equivalent with $H^{[s]+1/2}(I)$.

In the sequel we will write $(\cdot,\cdot)_\theta$ instead of $(\cdot,\cdot)_{\theta,2}$.

It is possible to interpolate between the spaces $W^{s_i}(\mu_i, \nu_i)$, $i = 1,2$,

and $\mathring{W}^{s_i}(\mu_i,\nu_i)$, $i = 1,2$.

Theorem 2.2. Let $s_1 \geq 0$, $s_2 \geq 0$, $s_1 \neq s_2$, $\nu_i \geq \mu_i + 2s_i$, $i = 1,2$, $0 < \theta < 1$, and

a) $(\mu_1-\nu_1)s_2 = (\mu_2-\nu_2)s_1$,

b) $s = (1-\theta)s_1 + \theta s_2$, $s \neq$ an integer,

c) $\nu = (1-\theta)\nu_1 + \theta\nu_2$,

d) $\dfrac{\mu-\nu}{s} = \dfrac{\mu_1-\nu_1}{s_1} = \dfrac{\mu_2-\nu_2}{s_2}$ (in the case that $s_1 = 0$ and $s_2 > 0$, one sets $\mu_1 = \nu_1$ and uses $\dfrac{\mu-\nu}{s} = \dfrac{\mu_2-\nu_2}{s_2}$).

Then

$$(W^{s_1}(\mu_1,\nu_1),\ W^{s_2}(\mu_2,\nu_2))_\theta = W^s(\mu,\nu). \tag{2.3}$$

For the proof, see Theorem 3.4.2 of [14]. □

We will now apply Theorems 2.1 and 2.2 in special cases we will be interested in later. Consider $\mathring{W}^{s_i}(\mu_i,\nu_i)$, $i = 1,2$, with

$$s_1 = 0,\ s_2 = r,\ r \geq 1 \text{ an integer}$$

$$\mu_1 = \nu_1 = \nu, \quad 0 < \nu < 1$$

$$\mu_2 = \nu - r,\ \nu_2 = \nu.$$

Then by Theorem 2.1

$$\mathring{W}^0(\mu_1,\nu_1) = W^0(\bar{\mu}_1,\bar{\nu}_1)$$

$$\mathring{W}^r(\mu_2,\nu_2) = W^r(\bar{\mu}_2,\bar{\nu}_2) = \mathring{W}^r(\bar{\mu}_2,\bar{\nu}_2)$$

where

$$\bar{\mu}_i = \mu_i,\ \bar{\nu}_i = \mu_i + 2s_i, \quad i = 1,2.$$

Further we have

$$0 = (\bar{\mu}_1 - \bar{\nu}_1)s_2 = (\bar{\mu}_2 - \bar{\nu}_2)s_1$$

and for $0 < \theta < 1$, $\theta r \neq$ an integer

$$\hat{s} = (1-\theta)s_1 + \theta s_2 = \theta r$$

$$\hat{\nu} = (1-\theta)\bar{\nu}_1 + \theta\bar{\nu}_2 = 2\hat{s} + [(1-\theta)\bar{\mu}_1 + \theta\bar{\mu}_2] = \hat{s} + \nu$$

$$\hat{\mu} = \hat{\nu} - 2\hat{s} = \nu - \hat{s}.$$

For $\{\hat{s}\} \neq 1/2$ and $\hat{\mu} + 2\hat{s} = \nu + \hat{s} = \hat{\nu} \neq 1 + 2k$, $k = 0, \ldots, [\hat{s}] - 1$ we have

$$\begin{aligned} \mathring{W}^{\hat{s}}(\hat{\mu}, \hat{\nu}) &= W^{\hat{s}}(\hat{\mu}, \hat{\mu}+2\hat{s}) \\ &= (W^0(\bar{\mu}_1, \bar{\nu}_1),\ W^r(\bar{\mu}_2, \bar{\nu}_2))_\theta \\ &= (\mathring{W}^0(\bar{\mu}_1, \bar{\nu}_1),\ \mathring{W}^r(\bar{\mu}_2, \bar{\nu}_2))_\theta \\ &= (\mathring{W}^0(\nu, \nu),\ \mathring{W}^r(\nu - r, \nu))_\theta \end{aligned}$$

and so

$$\mathring{W}^{\hat{s}}(\hat{\mu}, \tilde{\nu}) = (\mathring{W}(\nu, \nu),\ \mathring{W}^r(\nu - r, \nu))_\theta$$

for $\tilde{\nu} \leq \hat{\nu} = \theta r + \nu$.

Assume now that $0 < \nu < 1$, $\nu \neq 1/2$, $r = 1$, and $\theta = \nu$. Then we have $\hat{s} = \theta$, $\hat{\mu} = 0$ and hence

$$(\mathring{W}^0(\nu, \nu),\ \mathring{W}^1(\nu - 1, \nu))_\nu = W^\nu(0, 2\nu) = \mathring{W}^\nu(0, 0) = \mathring{H}^\nu(I).$$

We see that by interpolating the weighted Sobolev spaces $\mathring{W}^{s_i}(\mu_i, \nu_i)$, $i = 1, 2$, we can obtain the standard fractional Sobolev spaces.

2.3. <u>Polynomial bases on</u> I.

Let us create (eg. by Gramm Schmidt procedure) the orthogonal polynomial basis in the space $W^0(\alpha,\alpha)$. Denote this set of polynomials by $\{P_n(x;\alpha)\}$ where n is the degree of the polynomial.

<u>Theorem 2.3</u>. Let $\alpha < 1$. Then $P_n(x;\alpha)$ are the Jacobi polynomials with the index $\beta = -\alpha$, i.e., $P_n(x;\alpha) = P_n(x;-\beta,-\beta)$ where $P_n(x;-\beta,-\beta)$ is the standard Jacobi polynomial.

The theorem follows immediately from the basis properties of Jacobi polynomials, see eg. [16], [17]. □

We will list the major properties of the Jacobi polynomials

$$P_n(-1,\alpha) = (-1)^n P_n(1;\alpha) = (-1)^n \frac{\Gamma(-\alpha+n+1)}{n!\Gamma(-\alpha+1)} \tag{2.4a}$$

$$\int_{-1}^{1} (1-x^2)^{-\alpha} P_n(x;\alpha) P_m(x;\alpha)dx = 0 \quad \text{for} \quad n \neq m, \tag{2.4b}$$

$$\int_{-1}^{1} (1-x^2)^{-\alpha} (P_n(x;\alpha))^2 dx = \frac{2^{-2\alpha+1}\Gamma^2(-\alpha+n+1)}{n!(-2\alpha+2n+1)\Gamma(-2\alpha+n+1)} = A_n^2(\alpha) \tag{2.4c}$$

$$P_n'(x;\alpha) = \frac{1}{2}(-2\alpha+n+1)P_{n-1}(x;\alpha-1) \tag{2.4d}$$

$$(1-x^2)P_n''(x;\alpha) - 2(-\alpha+1)xP_n'(x;\alpha) + n(-2\alpha+n+1)P_n(x;\alpha) = 0 \tag{2.4e}$$

or

$$((1-x^2)^{-\alpha+1}P_n'(x;\alpha))' + n(-2\alpha+n+1)(1-x^2)^{-\alpha}P_n(x;\alpha) = 0.$$

We will also define the orthonormal Jacobi polynomials $\hat{P}_n(x;\alpha)$

$$\hat{P}_n(x;\alpha) = P_n(x;\alpha)A_n^{-1}(\alpha) \tag{2.5a}$$

and get

$$\hat{P}_n(+1;\alpha) = (-1)^n\hat{P}_n(-1,\alpha) = d_n(\alpha)n^{-\alpha+(1/2)}, \tag{2.5b}$$

where $d_n(\alpha)$, $n = 0,1,2,\ldots$ is bounded from above and below by constants depending only on α.

So far we assumed that $\alpha < 1$. Let us address now the case $\alpha = 1$. Obviously a polynomial $P(x)$ belongs to $W(1,1)$ only when $P(\pm 1) = 0$.

Theorem 2.4. Let $\alpha = 1$. Then the orthogonal polynomial basis $\{P_n(x;1)\}$ in $W(1,1)$ is given by

$$P_n(x;1) = \frac{n-1}{2}\int_{-1}^{x} P_{n-1}(t;0)dt, \quad n \geq 2 \tag{2.6}$$

and (2.4b), (2.4c) hold for $n \geq 2$.

Proof. $P_n(t;0)$ is the Legendre polynomial. Because of the orthogonality of Legendre polynomials $P_n(\pm 1;1) = 0$, $n \geq 2$. Obviously $P_n(x;1)$ belongs to $W^0(1,1)$. We have to show that

$$\int_{-1}^{1}(1-x^2)^{-1}P_n(x;1)P_m(x;1)dx = 0$$

for $n,m \geq 1$, $n \neq m$. For $\alpha = 0$ we have from (2,4e)

$$P_n(x;0) = -((1-x^2)P_n'(x;0))'\ \frac{1}{(n+1)n}$$

and hence

$$P_n(x;1) = -\frac{1}{2n}(1-x^2)P_{n-1}'(x;0).$$

Therefore

$$\int_{-1}^{1}(1-x^2)^{-1}P_n(x;1)P_m(x;1)dx = C(m,n)\int_{-1}^{1}(1-x^2)P_{n-1}'(x;0)P_{m-1}'(x;0)dx$$

$$= C_1(m,n)\int_{-1}^{1}(1-x^2)P_{n-2}(x;-1)P_{m-2}(x;-1)dx$$

$$= 0,$$

when using (2.4b,d) for $m \neq n$, $n,m \geq 2$. It is easy to check that (2.4c) holds for $\alpha = 1$ and $n \geq 2$. □

Using now (2.4c) and (2.4d) we see that the system $\{P_n(x;\alpha)\}$ is an orthogonal base in $W^1(\alpha-1,\alpha)$ for $\alpha < 1$ and of $\mathring{W}^1(\alpha-1,\alpha)$ for $\alpha = 1$.

2.4. The approximation properties of the polynomials on I.

Let $\hat{P}(x;\alpha)$, $\alpha \leq 1$, be the set of the orthonormal polynomials in $W^0(\alpha,\alpha)$. Assuming $u \in W^0(\alpha,\alpha)$, $\alpha \leq 1$, we can write

$$u(x) = \sum_{k=0}^{\infty} c_k \hat{P}_k(x;\alpha) \tag{2.7}$$

and the series (2.7) converges in $W^0(\alpha,\alpha)$.

Let us define

$$\mathcal{P}_p^{\alpha} u = \sum_{k=0}^{p} c_k \hat{P}_k(x;\alpha) \tag{2.8}$$

and

$$Q_p^{\alpha} u = u - \mathcal{P}_p^{\alpha} u = \sum_{p+1}^{\infty} c_k \hat{P}_k(x;\alpha). \tag{2.9}$$

We have

$$\|\hat{P}_k'(x;\alpha)\|_{W^0(\alpha-1,\alpha-1)} = A_k^{-1}(\alpha)\|P_k'(x;\alpha)\|_{W^0(\alpha-1,\alpha-1)}$$

$$= A_k^{-1}(\alpha)A_{k-1}(\alpha-1)\tfrac{1}{2}(-2\alpha+k+1)\|\hat{P}_{k-1}(x;\alpha-1)\|_{W^0(\alpha-1,\alpha-1)}$$

$$= A_k^{-1}(\alpha)A_k(\alpha-1)\tfrac{1}{2}(-2\alpha+k+1) = B_k(\alpha)\cdot k$$

with

$$0 < C_1(\alpha) \le B_k(\alpha) \le C_2(\alpha) < \infty$$

so that

$$\hat{P}_k'(x;\alpha) = \hat{P}_{k-1}(x;\alpha-1)kD_k(\alpha)$$

where $D_k(\alpha)$ is bounded from above and below by the constant dependent only on α.

Assume now that $u \in W^1(\alpha-1,\alpha)$ then $u' \in W^0(\alpha-1,\alpha-1)$. For u given by (2.7) we have now

$$u' = \sum_{k=0}^{\infty} c_k k D_k \hat{P}_{k-1}(x;\alpha-1) \tag{2.10}$$

and hence

$$c_1(\alpha)\|u\|^2_{W^1(\alpha-1,\alpha)} \le \sum_{k=0}^{\infty} c_k^2(k^2+1) \le c_2(\alpha)\|u\|^2_{W^1(\alpha-1,\alpha)}. \tag{2.11a}$$

Similarly for any integer $r \ge 0$ and $u \in W^r(\alpha-r,\alpha)$ given by (2.7) we have

$$C_1(\alpha,r)\|u\|^2_{W^r(\alpha-r,\alpha)} \le \sum_{k=0}^{\infty} c_k^2(k^{2r}+1) \le C_2(\alpha,r)\|u\|^2_{W^r(\alpha-r,\alpha)}. \tag{2.11b}$$

(2.11b) leads immediately to

<u>Theorem 2.5</u>. Let r,s be integers and $r \ge s \ge 0$, $u \in W^r(\alpha-r,\alpha)$, $\alpha \le 1$. Then

$$\|Q_p^\alpha u\|_{W^s(\alpha-s,\alpha)} \le C(\alpha,r,s)p^{-(r-s)}\|u\|_{W^r(\alpha-r,\alpha)}. \qquad \square$$

So far we have assumed $u \in W^r(\alpha-r,\alpha)$, $s \le r$ and we have

$\mathcal{P}_p^\alpha u \in W^s(\alpha-s,\alpha)$. Let us now consider the case when $u \in \mathring{W}^r(\alpha-r,\alpha)$. Then obviously in general $\mathcal{P}_p^\alpha u \notin \mathring{W}^1(\alpha-1,\alpha)$.

<u>Lemma 2.6</u>. Let $n > 0$ an integer, $\alpha < 1$, and

(2.12a)
$$\psi_{1,n}^{[\alpha]}(x) = \sum_{k=0}^{[n/2]} (2k+1)\hat{P}_{2k}(x;\alpha)$$

(2.12b)
$$\psi_{2,n}^{[\alpha]}(x) = \sum_{k=0}^{[(n-1)/2]} (2k+1)\hat{P}_{2k+1}(x;\alpha).$$

Then $\psi_{i,n}^{[\alpha]}(x)$ have following properties:

a) $\psi_{1,n}^{[\alpha]}(x)$ is an even function and $\psi_{2,n}^{\alpha}(x)$ is an odd function;

b) $\psi_{i,n}^{[\alpha]}(x)$, $i = 1,2$ is a polynomial of degree n;

c) (2.13a) $c_1(\alpha)n^{-\alpha+(5/2)} \le \psi_{i,n}^{[\alpha]}(1) \le c_2(\alpha)n^{-\alpha+(5/2)}$, $i = 1,2$

d) (2.13b) $d_1(\alpha)n^{(3/2)+r} \le \|\psi_{i,n}^{[\alpha]}\|_{W^r(\alpha-r,\alpha)} \le d_2(\alpha)n^{(3/2)+r}$,

$r = 0,1$, $i = 1,2$.

The above statements follow immediately from the basic properties of Jacobi polynomials. □

Let us now define

$$\mathring{\mathcal{P}}_p^\alpha u = \mathcal{P}_p^\alpha u - \frac{1}{2}\sum_{i=1}^{2}((\mathcal{P}_p^\alpha u)(1)+(-1)^{i+1}(\mathcal{P}_p^\alpha u)(-1))\frac{\psi_{i,p}^{[\alpha]}(x)}{\psi_{i,p}^{[\alpha]}(1)}$$

and

$$\mathring{\mathcal{P}}_p^{\circ\alpha} u = \mathcal{P}_p^\alpha u - \frac{1}{2}\sum_{i=1}^{2}((\mathcal{P}_p^\alpha u)(1)+(-1)^{i+1}(\mathcal{P}_p^\alpha u)(-1))\frac{\psi_{i,1}^{[\alpha]}(x)}{\psi_{i,1}^{[\alpha]}(1)}.$$

Then obviously $(\mathring{\mathcal{P}}_p^\alpha u)(\pm 1) = (\mathring{\mathcal{P}}_p^{\circ\alpha} u)(\pm 1) = 0$ and $\mathring{\mathcal{P}}_p^\alpha u$, $\mathring{\mathcal{P}}_p^{\circ\alpha} u \in W^r(\alpha-r,\alpha)$, $r =$

0, 1. Further, let $\mathring{Q}_p^{\alpha}u = u - \mathring{\mathcal{P}}_p^{\alpha}u$ and $\mathring{Q}_p^{\circ\alpha}u = u - \mathring{\mathcal{P}}_p^{\circ\alpha}u$.

<u>Theorem 2.7</u>. Let $0 < \alpha \le 1$, $r = 0, 1$, s an integer, $s \ge r$, $\alpha > 1 - s$. Then

$$\|\mathring{Q}_p^{\alpha}u\|_{\mathring{W}^r(\alpha-r,\alpha)} \le C(\alpha, r, s)p^{-(s-r)}\|u\|_{\mathring{W}^s(\alpha-s,\alpha)}. \tag{2.14}$$

<u>Proof</u>. Because of the definition of $\mathring{W}^s(\alpha-s,\alpha)$ it is sufficient to prove (2.14) for $u \in C_0^{\infty}$. Let $s \ge 1$, $1 > \alpha > 1 - s$

$$u = \sum_{k=0}^{\infty} c_k \hat{P}_k(x;\alpha)$$

and

$$u(\pm 1) = 0.$$

Hence

$$\begin{aligned} |(\mathcal{P}_p^{\alpha}u)(\pm 1)| &\le \sum_{k=p+1}^{\infty} |c_k||\hat{P}_k(\pm 1;\alpha)| \\ &\le C\sum_{k=p+1}^{\infty} |c_k|k^{-\alpha+(1/2)} \le C\sum_{k=p+1}^{\infty} |c_k|k^s k^{-\alpha+(1/2)-s} \\ &\le C\left(\sum_{k=p+1}^{\infty} |c_k|^2 k^{2s}\right)^{1/2} p^{-\alpha+1-s} \\ &\le C\|u\|_{\mathring{W}^s(\alpha-s,\alpha)} p^{-\alpha+1-s}. \end{aligned} \tag{2.15}$$

Therefore, using Theorem 2.5 and Lemma 2.6 we get

$$\begin{aligned} \|\mathring{Q}_p^{\alpha}u\|_{W^r(\alpha-r,\alpha)} &\le \|Q_p^{\alpha}u\|_{W^r(\alpha-r,\alpha)} \\ &\quad + C\|u\|_{W^s(\alpha-s,\alpha)} p^{-\alpha+1-s} \frac{1}{p^{-\alpha+(5/2)}} p^{(3/2)+r} \end{aligned}$$

$$\le Cp^{-(s-r)}\|u\|_{\mathring{W}^s(\alpha-s,\alpha)}.$$

Hence, with $r = 1$,

$$\int_{-1}^{+1}\left[(1-x^2)^{-(\alpha-1)}[(\mathring{Q}_p^\alpha u)']^2 + (1-x^2)^{-\alpha}(Q_p^\alpha u)^2\right]dx \le Cp^{-(s-1)}\|u\|_{\mathring{W}^s(\alpha-s,\alpha)}$$

and $(\mathring{Q}_p^\alpha u)(\pm 1) = 0$. Using Theorem 330 of [19] we get for $\alpha > 0$

$$\int_{-1}^{+1}(1-x^2)^{-(1+\alpha)}(\mathring{Q}_p^\alpha u)^2dx \le C\int_{-1}^{+1}(1-x^2)^{-(\alpha-1)}[(\mathring{Q}_p^\alpha u)']^2dx$$

which shows that also

$$\|\mathring{Q}_p^\alpha u\|_{\mathring{W}^1(\alpha-1,\alpha)} \le C\|\mathring{Q}_p^\alpha u\|_{W^1(\alpha-1,\alpha+1)} \le Cp^{-(s-1)}\|u\|_{\mathring{W}^s(\alpha-s,\alpha)}$$

and (2.14) is proven for $r = 1$, $s \ge 1$.

Next let $u \in W^0(\alpha,\alpha) \cap C_0^\infty$, $\alpha < 1$. Then

$$|(\mathcal{P}_p^\alpha u)(\pm 1)| \le \sum_0^p |c_k||\hat{P}_k(\pm 1;\alpha)| \le C\sum_{k=0}^p |c_k|k^{-\alpha+(1/2)} \le C\|u\|_{W^0(\alpha,\alpha)}p^{-\alpha+1}$$

and hence

$$\|\mathring{Q}_p^\alpha u\|_{W^0(\alpha,\alpha)} \le \|Q_p^\alpha u\|_{W^0(\alpha,\alpha)} + C\|u\|_{W^0(\alpha,\alpha)}\left[p^{-\alpha+1}\frac{1}{p^{-\alpha+(5/2)}}p^{3/2}\right] \le C\|u\|_{W^0(\alpha,\alpha)}.$$

Because $\mathcal{P}_p^\alpha u \in \mathring{W}^1(0,\alpha)$ for $\alpha = 1$ the theorem also holds for $\alpha = 1$ when $\mathring{\mathcal{P}}_p^1 = \mathcal{P}_p^1$. □

Theorem 2.7, together with Theorems 2.1 and 2.2, leads to a series of important approximation results. As we have seen earlier

$$\hat{\mathring{W}}^s(\hat{\mu},\tilde{\nu}) = (\mathring{W}^0(\alpha,\alpha),\mathring{W}^r(\alpha-r,\alpha))_\theta$$

where

$$\hat{s} = \theta r, \ \hat{s} \neq \text{integer}, \ \{s\} \neq 1/2$$

$$\hat{\mu} = \alpha - \theta r$$

$$\hat{\mu} + 2\hat{s} = \hat{\nu} = \hat{s} + \alpha \neq 1 + 2k, \quad k = 0, \dots, [\hat{s}] - 1$$

$$\tilde{\nu} \le \hat{\nu}.$$

Hence by Theorem 2.7 we get for $0 < \theta_1 < 1$, $\theta_1 \neq 1/2$, $s \ge 1$, an integer, $\alpha \le 1$, $\alpha > 1 - s$, $(r = 1)$

$$\|\mathring{Q}_p^\alpha u\|_{\mathring{W}^{\theta_1}(\hat{\mu}_{\theta_1}, \hat{\nu}_{\theta_1})} \le Cp^{-(s-\theta_1)} \|u\|_{\mathring{W}^s(\alpha - s, \alpha)} \tag{2.16}$$

where

$$\hat{\mu}_{\theta_1} = \alpha - \theta_1$$

$$\hat{\nu}_{\theta_1} = \alpha + \theta_1$$

and (2.16) holds (only) under the assumption that $\theta_1 \neq 1/2$, $s \ge 1$ an integer.

Let us remark that we can replace on the left hand side of (2.16) the norm of $\mathring{W}^{\theta_1}(\alpha-\theta_1, \alpha+\theta_1)$ by the norm of the space $\mathring{W}^{\theta_1}(\alpha-\theta_1, \alpha)$. Let now $0 < \theta < 1$, $\theta \neq 1/2$

$$s_\theta = [s_\theta] + \theta.$$

If $[s] \ge 1$ then we can use the interpolation on the right hand side and get

$$\|\mathring{Q}_p^\alpha u\|_{\mathring{W}^{\theta_1}(\alpha-\theta_1, \alpha)} \le Cp^{-(s_\theta - \theta_1)} \|u\|_{\mathring{W}^{s_\theta}(\alpha - s_\theta, \alpha)}$$

where we have excluded the case

$$\alpha + s_\theta = 1 + 2k, \quad k = 0, \ldots, [s_\theta] - 1.$$

If $[s] = 0$ then first by simultaneous interpolation on both sides of (2.14) we prove that

$$\|\mathring{Q}_p^\alpha u\|_{\mathring{W}^{\theta_1}(\alpha-\theta_1,\alpha)} \le C\|u\|_{\mathring{W}^{\theta_1}(\alpha-\theta_1,\alpha)}$$

and then using (2.16) for $s = 1$ by interpolating on the right hand side we get

$$\|\mathring{Q}_p^\alpha u\|_{\mathring{W}^{\theta_1}(\alpha-\theta_1,\alpha)} \le Cp^{-(\theta-\theta_1)}\|u\|_{\mathring{W}^{\theta}(\alpha-\theta,\alpha)}$$

provided that $\theta \ge \theta_1$ and $\theta, \theta_1 \ne 1/2$. We have

<u>Theorem 2.8</u>. Let $0 < \alpha \le 1$, $0 < s_1 \le 1$, $s_2 > s_1$, $s_i \ne [s_i] + 1/2$, $\alpha_i + s_i \ne 1 + 2k$, $k = 0, \ldots, [s_i] - 1$. Then

$$\|\mathring{Q}_p^\alpha u\|_{\mathring{W}^{s_1}(\alpha-s_1,\alpha)} \le Cp^{-(s_2-s_1)}\|u\|_{\mathring{W}^{s_2}(\alpha-s_2,\alpha)}. \tag{2.17}$$

In addition we can replace the spaces $\mathring{W}^{s_i}(\alpha-s_i,\alpha)$ by $\mathring{W}^{s_i}(\alpha-s_i,\alpha+s_i)$ in (2.17) and also $\mathring{W}^{s_1}(\alpha-s_1,\alpha)$ by $\mathring{W}^{s_1}(\beta,\gamma)$ where $\beta \le \alpha - s_1$, $\gamma \le \alpha$ and $\mathring{W}^{s_2}(\alpha-s_2,\alpha)$ by $\mathring{W}^{s_2}(\bar\beta,\bar\gamma)$ with $\bar\beta \ge \alpha - s_2$ and $\bar\gamma \ge \alpha$. □

(Because of direct use of (2.14) we do not exclude the case of s_i being integers).

Select now $\alpha = s_1$, $0 < \alpha \le 1$, $\alpha \ne 1/2$. Then we get from (2.17)

$$\|\mathring{Q}_p^\alpha u\|_{\mathring{W}^{\alpha}(0,0)} = \|\mathring{Q}_p^\alpha u\|_{\mathring{H}^{\alpha}(I)} \le Cp^{-(s_2-\alpha)}\|u\|_{\mathring{W}^{s_2}(\alpha-s_2,\alpha)}. \tag{2.18a}$$

Select now $s_1 < \alpha$, $s_1 \ne 1/2$ in (2.17):

$$\|\mathring{Q}_p^\alpha u\|_{\mathring{H}^{s_1}(I)} \le \|\mathring{Q}_p^\alpha u\|_{\mathring{W}^{s_1}(0,0)} = \|\mathring{Q}_p^\alpha u\|_{\mathring{W}^{s_1}(\alpha-s_1,\alpha)} \tag{2.18b}$$

$$\le Cp^{-(s_2-s_1)}\|u\|_{\mathring{W}^{s_2}(\alpha-s_2,\alpha)}.$$

For $s_1 > \alpha$ we obviously cannot use the argument above. In (2.18) we excluded the cases $s_i \ne [s]+1/2$ and $\alpha+s_i \ne 1+2k$, $k = 1,\dots,[s_i]-1$.

In the next section we will be especially interested in approximation in the space $\mathring{H}^{1/2}(I)$ which was excluded from our consideration. Hence we will consider $\mathring{H}^{(1/2)+\varepsilon}(I)$ and get from (2.18a) the following.

Theorem 2.9. Let $\frac{1}{2} \ge \varepsilon > 0$, $\alpha = \frac{1}{2}+\varepsilon$, $s > \alpha$. Then

$$\|\mathring{Q}_p^{\alpha}u\|_{\mathring{H}^{(1/2)+\varepsilon}(I)} \le Cp^{-(s-(1/2)-\varepsilon)}\|u\|_{\mathring{W}^{s}(\alpha-s,\alpha)} \tag{2.19}$$

for $\alpha+s \ne 1+2k$, $k = 0,\dots,[s]-1$. □

On the left hand side of (2.19), we can replace $\mathring{H}^{(1/2)+\varepsilon}(I)$ by $H_{00}^{1/2}(I)$ and on the right hand side $\mathring{W}^{s}(\alpha-s,\alpha)$ by $\mathring{W}^{s}(\beta,\gamma)$, $\beta \ge \alpha-s$, $\gamma \ge \alpha$.

Remark 2.1. Using Theorem 2.9 we have lost p^{ε} in the estimate of the error in $H_{00}^{1/2}(I)$. This case can be studied separately. For example in [11] we have shown that for $s > 1/2$

$$\|\mathring{Q}_p^{\circ 1/2}u\|_{H_{00}^{1/2}(I)} \le Cp^{-(s-(1/2))}(\log p)^{1/2}\|u\|_{\mathring{W}^{s}(0,0)}.$$

So far we have addressed the approximation properties of $\mathring{Q}_p^{\alpha}$. Let us now analyze $\mathring{Q}_p^{\circ\alpha}$. We have

$$\mathring{\mathcal{P}}_p^{\circ\alpha}u - \mathring{\mathcal{P}}_p^{\alpha}u = \frac{1}{2}\sum_{i=1}^{2}((\mathcal{P}_p^{\alpha}u)(1) + (-1)^{i+1}(\mathcal{P}_p^{\alpha}u)(1))\left[\frac{\psi_{i,p}^{[\alpha]}(x)}{\psi_{i,p}^{[\alpha]}(1)} - \frac{\psi_{i,1}^{[\alpha]}(x)}{\psi_{i,1}^{[\alpha]}(1)}\right].$$

By (2.15), we have for $s > -\alpha+1$ as before

$$|(\mathcal{P}_p^{\alpha}u)(\pm 1)| \le Cp^{-\alpha+1-s}\|u\|_{\mathring{W}^{s}(\alpha-s,\alpha)}.$$

Remembering that $(\mathcal{P}^{\circ\alpha}_p u - \mathcal{P}^{\alpha}_p u)(\pm 1) = 0$ using Theorem 2.1 and also Theorem 330 of [19] we get for $\alpha > 0$

$$\|(\mathcal{P}^{\circ\alpha}_p u - \mathcal{P}^{\alpha}_p u)\|_{W^1(\alpha-1,\alpha)} \geq C\|(\mathcal{P}^{\circ\alpha}_p u - \mathcal{P}^{\alpha}_p u)\|_{W^1(\alpha-1,\alpha+1)}$$
$$= C\|\mathcal{P}^{\circ\alpha}_p u - \mathcal{P}^{\alpha}_p u)\|_{\mathring{W}^1(\alpha-1,\alpha)}.$$

Hence by interpolation we get for $s_2 > s_1$, $0 \leq s_1 \leq 1$,

$$\|\mathcal{P}^{\circ\alpha}_p u - \mathcal{P}^{\alpha}_p u\|_{\mathring{W}^{s_1}(\alpha-s_1,\alpha)} \leq Cp^{-\alpha+1-s_2}[1+p^{\alpha-(5/2)}p^{(3/2)+s_1}]\|u\|_{\mathring{W}^{s_2}(\alpha-s_2,\alpha)}$$
$$\leq Cp^{-(s_2-s_1)}\|u\|_{\mathring{W}^{s_2}(\alpha-s_2,\alpha)}$$

provided $-\alpha+1 \leq s_1$ and as before $\alpha+s_2 \neq 1+2k$, $k = 0,\dots,[s_2]-1$. Let us summarize our results in

<u>Theorem 2.10</u>. Let $0 < \alpha \leq 1$, $s_i \geq 0$, $s_i+\alpha \neq 1+2k$, $k = 0,\dots,[s]-1$, $i = 1,2$. Then

1) for $s_2 > s_1$, $s_i \neq [s_i]+\frac{1}{2}$

(2.20a) $$\|\mathcal{Q}^{\alpha}_p u\|_{\mathring{W}^{s_1}(\alpha-s_1,\alpha)} \leq Cp^{-(s_2-s_1)}\|u\|_{\mathring{W}^{s_2}(\alpha-s_2,\alpha)}.$$

If in addition $-\alpha+1 \leq s_1$ then

(2.20b) $$\|\mathcal{Q}^{\circ\alpha}_p u\|_{\mathring{W}^{s_1}(\alpha-s_1,\alpha)} \leq Cp^{-(s_2-s_1)}\|u\|_{\mathring{W}^{s_2}(\alpha-s_2,\alpha)}.$$

2) For $0 \leq \beta \leq \alpha \leq 1$, $s_2 > \beta$, $\beta \neq 1/2$

(2.20c) $$\|\mathcal{Q}^{\alpha}_p u\|_{\mathring{H}^{\beta}(I)} \leq Cp^{-(s_2-\beta)}\|u\|_{\mathring{W}^{s_2}(\alpha-s_2,\alpha)}.$$

If in addition $-\alpha+1 \leq \beta < s_2$, then

(2.20d) $$\|\mathring{Q}_p^{\circ\alpha} u\|_{\mathring{H}^\beta(I)} \le Cp^{-(s_2-\beta)} \|u\|_{\mathring{W}^{s_2}(\alpha - s_2, \alpha)} . \qquad \square$$

It was essential in (2.20c) (2.20d) that $\beta \le \alpha$. The case $\alpha < \beta$ has to be studied separately. Let us mention first

Lemma 2.11. Let u be a polynomial of degree p and $0 \le \alpha \le \beta$. Then

$$\|u\|_{H^\beta(I)} \le Cp^{2(\beta-\alpha)} \|u\|_{H^\alpha(I)} . \qquad \square$$

For the proof, see eg. Lemma 2.4 of [18] or [1].

Now we can prove

Theorem 2.12. Let $0 < \alpha \le \beta \le 1$, $\alpha, \beta \neq 1/2$, $\beta < s_2$, $s_2 \neq [s_2] + \frac{1}{2}$, $s_2 + \alpha \neq 1 + 2k$, $k = 0, \ldots, [s_2] - 1$. Then

(2.21a) $$\|\mathring{Q}_p^{\alpha} u\|_{\mathring{H}^\beta(I)} \le Cp^{-(s_2-\beta)+(\beta-\alpha)} \|u\|_{\mathring{W}^{s_2}(\beta - s_2, \beta)} .$$

If in addition $1 - \beta \le \alpha \le s_2$ then

(2.21b) $$\|\mathring{Q}_p^{\circ\alpha} u\|_{\mathring{H}^\beta(I)} \le Cp^{-(s_2-\beta)+(\beta-\alpha)} \|u\|_{\mathring{W}^{s_2}(\beta - s_2, \beta)} .$$

Proof. Using (2.20c) we get

$$\|\mathring{Q}_p^{\alpha} u\|_{\mathring{H}^\alpha(I)} \le Cp^{-(s_2-\alpha)} \|u\|_{\mathring{W}^{s_2}(\alpha - s_2, \alpha)} ,$$

$$\|\mathring{Q}_p^{\beta} u\|_{\mathring{H}^\alpha(I)} \le Cp^{-(s_2-\alpha)} \|u\|_{\mathring{W}^{s_2}(\beta - s_2, \beta)} ,$$

and hence

$$\|\mathring{Q}_p^{\alpha} u - \mathring{Q}_p^{\beta} u\|_{\mathring{H}^\alpha(I)} = \|\mathcal{P}_p^{\alpha} u - \mathcal{P}_p^{\beta} u\|_{H^\alpha(I)} \le Cp^{-(s_2-\alpha)} \|u\|_{\mathring{W}^{s_2}(\beta - s_2, \beta)} .$$

Using Lemma 2.11 we get

$$\|\mathring{Q}_p^{\alpha}u - \mathring{Q}_p^{\beta}u\|_{H^{\beta}(I)} \le Cp^{2(\beta-\alpha)-(s_2-\alpha)}\|u\|_{\mathring{W}^{s_2}(\beta-s_2,\beta)}$$

and hence

$$\|\mathring{Q}_p^{\alpha}u\|_{\mathring{H}^{\beta}(I)} \le Cp^{-(s_2-\beta)+(\beta-\alpha)}\|u\|_{\mathring{W}^{s_2}(\beta-s_2,\beta)}$$

where we wrote on the left hand side $\mathring{H}^{\beta}(I)$ instead of $H^{\beta}(I)$ because $\beta \neq 1/2$, $\mathring{Q}_p^{\beta}u \in \mathring{H}^{\beta}(I)$ and $\mathring{Q}_p^{\alpha}u - \mathring{Q}_p^{\beta}u$ is a polynomial vanishing at $x = \pm 1$. The second part of the theorem follows easily, too. □

Remark 2.2. Using Theorem 2.12, we see that for $\alpha > 1/2$, $s \ge (1/2) + 2\varepsilon$, $\varepsilon > 0$, we have

$$\|\mathring{Q}_p^{\alpha}u\|_{H_{00}^{1/2}} \le C(\varepsilon)p^{-(s-(1/2))+\varepsilon}\|u\|_{\mathring{W}^{s}(\alpha-s,\alpha)}$$

and the same estimate holds also for $\mathring{Q}_p^{\circ\alpha}$.

In the next section we will especially be interested in the approximation in the space $H_{00}^{1/2}$, which case was excluded from our consideration. We addressed this case only via the approximation in $\mathring{H}^{(1/2)+\varepsilon}(I)$.

It is well known that $\{P_k(x;1/2)\}$ are Tchebyshev polynomials which can be written in the form

$$P_k(x;1/2) = \cos(k \arccos x)$$

$$\hat{P}_k(x;1/2) = \sqrt{\frac{2}{\pi}} \cos(k \arccos x).$$

We have shown in [11] that if

$$u = \sum_{k=0}^{\infty} c_k P_k(x;1/2)$$

then

$$(2.22) \qquad \|u\|^2_{H^{1/2}(I)} = \sum_{k=0}^{\infty} |c_k|^2 k + |c_0|^2 .$$

The space $H_{00}^{1/2}(I)$ then has a norm which is equivalent to

$$(2.23) \qquad \|u\|^2_{H_{00}^{1/2}(I)} = \int_{-1}^{1} (1-x^2)^{-1} u^2 dx + \|u\|^2_{H^{1/2}(I)} .$$

This norm is equivalent to $\|\cdot\|_{\mathring{W}^{1/2}(0,1)}$. Obviously $\mathcal{P}_p^{1/2}u$ and $\mathcal{P}_p^{\circ 1/2}u \in H_{00}^{1/2}(I)$ if $u \in \mathring{H}^k(I)$, $k > 1/2$.

In [11], we have proven various properties of $\mathcal{P}_p^{\circ 1/2}$ for example

$$\|\mathring{Q}_p^{\circ 1/2}u\|_{H_{00}^{1/2}(I)} \le Cp^{-(k-(1/2))}(\log^{1/2} p)\|u\|_{\mathring{H}^k(I)} \qquad k > 1/2.$$

Let us remark that in general

$$\|\mathring{Q}_p^{1/2}u\|_{H_{00}^{1/2}(I)} \quad \text{and} \quad \|\mathring{Q}_p^{\circ 1/2}u\|_{H_{00}^{1/2}(I)}$$

(and analogously in other cases) are not necessarily monotonic non-increasing functions of p. Of course, if we define $\mathcal{P}_p^{1/2}$ as the projection onto the set of polynomials of degree $\le p$ in the norm (2.23), then the monotonicity would be guaranteed. If we neglect the first term in (2.23) and define $\mathcal{P}_p^{1/2}$ as a projection in this norm, then we get a monotone sequence but now $\mathcal{P}_p^{1/2} \notin H_{00}^{1/2}$.

2.5. Numerical experimentation.

In the next section we will analyze the error of the finite element method for two dimensional elliptic problems. This analysis leads naturally to the measure of the error on I in the norm $\|\cdot\|_{H^{1/2}(I)}$, respectively

$\|\cdot\|_{H_{00}^{1/2}(I)}$. Hence we will be interested here in the computational analysis of the error in the one-dimensional case, in the norm $\|\cdot\|_{H^{1/2}(I)}$. More precisely, we will analyze $\|u_\omega - \mathcal{P}_p^\alpha u_\omega\|_{H^{1/2}(I)}$ where

$$u_\omega(x) = \frac{1}{\sqrt{1-2\omega x+x^2}} = \sum_{k=0}^{\infty} \omega^k P_k(x;0), \quad 0 < \omega < 1.$$

We choose the functions $u_\omega(x)$ since they are characteristic for applications. In practice, in the finite element method, the singularity is almost always located at an end point of I. If $\omega \longrightarrow 1$ then $u_\omega(x)$ becomes singular at $x = -1$. Hence the parameter ω characterizes the smoothness of u_ω when the nonsmooth behavior occurs at an end point of I.

Let

$$r_p^\alpha(\omega) = \frac{\|u_\omega - \mathcal{P}_p^\alpha u_\omega\|_{H^{1/2}(I)}}{\|u_\omega - \mathcal{P}_p^{1/2} u_\omega\|_{H^{1/2}(I)}}$$

where $\|\cdot\|_{H^{1/2}(I)}$ is given by (2.22). Obviously $r_p^\alpha(\omega) \geq 1$ and $r_p^\alpha(\omega)$ expresses the quality of the performance of $\mathcal{P}_p^\alpha$ in $\|\cdot\|_{H^{1/2}(I)}$. Figure 2.2 shows $r_p^\alpha(\omega)$ for $\omega = 0.9$, 0.95 and 0.98 and for the (extreme) cases $\alpha = 0,1$. Table 2.1 gives the values $r_p^0(0.9)$ and $r_p^1(0.9)$. The values of $r_p^1(0.95)$ and $r_p^1(0.98)$ are close together.

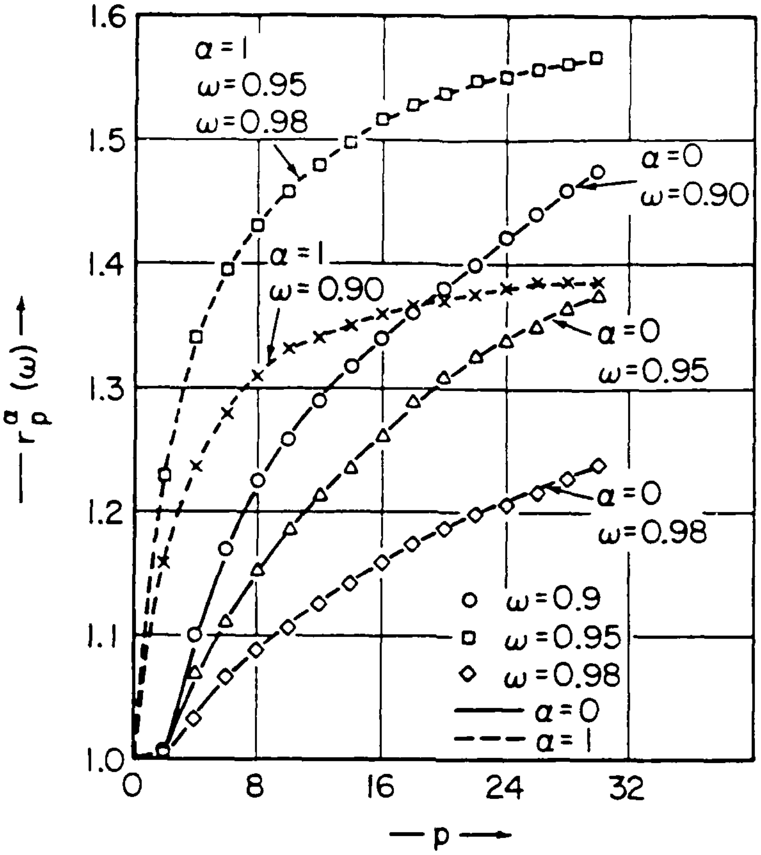

Figure 2.2. Comparison of the performance of $\mathcal{P}_p^0$ and $\mathcal{P}_p^1$.

Table 2.1.

The values $r_p^\alpha(0.9)$, $\alpha = 0, 1$

p	$r_p^0(0.9)$	$r_p^1(0.9)$
1	1.0000	1.0000
2	1.0113	1.1611
3	1.0904	1.2741
4	1.1067	1.2450
5	1.1516	1.3218
10	1.2602	1.3281
15	1.3192	1.3854
20	1.3892	1.3720
25	1.4078	1.4038
30	1.4663	1.3909
35	1.4676	1.4130
40	1.5207	1.4018
50	1.5622	1.4089

We see that $r_p^0(\alpha)$ decreases (in our range) as $\alpha \to 1$ while $r_p^1(\alpha)$ increases. For p large (with respect to $\frac{1}{1-\omega}$) the projection $\mathcal{P}_p^1$ performs better than $\mathcal{P}_p^0$, but for practical p ($p < 15$, say), $\mathcal{P}_p^0$ seems to be preferable to $\mathcal{P}_p^1$. As we will see in the next section a similar effect is present in the context of the finite element method.

3. The Finite Element Method

3.1. The model problem.

Let Ω be the domain as given in Section 2.1. Let $\mathcal{M} = \{1,\dots,M\}$ and $\mathcal{D} \subset \mathcal{M}$. Denote now $\Gamma^D = \bigcup_{i\in\mathcal{D}} \bar{\Gamma}_i$ and $\Gamma^N = \Gamma - \Gamma^D$. We will call Γ^D, respectively Γ^N the Dirichlet, respectively the Neumann boundary. Let $H = H^1(\Omega)$ and $H_0 = \{u \in H | u = 0 \text{ on } \Gamma^D\}$ and $B(u,v)$ be a continuous symmetric bilinear form on $H\times H$. We will assume that for all $u \in H_0$ we have

$$B(u,u) \geq \gamma\|u\|^2_{H}, \quad \gamma > 0.$$

Further, let F be a continuous linear functional on H and $g \in H^{1/2}(\Gamma^D)$, $g^{[i]} = g|_{\Gamma_i} \in H^k(\Gamma_i)$, $i \in \mathcal{D}$, $k > 1/2$.

Our model problem is now

$$(3.1) \qquad \begin{cases} \text{Find } u_0 \in H \text{ such that} \\ \quad u_0 = g \text{ on } \Gamma^D \\ \text{and} \\ B(u_0,v) = F(v), \quad \forall\, v \in H_0 \end{cases}$$

Our model problem has a unique solution.

3.2. The p-version of the finite element method.

Let $\bar{\Omega} = \bigcup_{i=1}^{q} \bar{\Omega}_i$, where Ω_i, $i = 1,\dots,q$ are open curved quadrilaterals or triangles called elements of the partition of Ω. The vertices of Ω_i are called the nodes of the partition. We will assume that the nodes which are located on the boundary Γ of Ω coincide with the vertices A_j of Ω as shown in Figure 3.1.

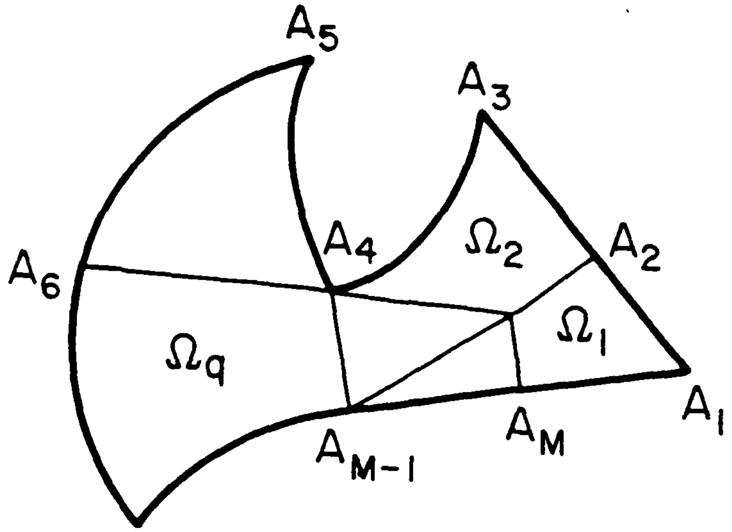

Figure 3.1. The scheme of the partitioned domain.

By $S = (-1,1)^2$ and $T = \{\xi,\eta \mid -1 < \xi \le 0,\ 0 < \eta < (\xi+1)\sqrt{3};\ 0 \le \xi < 1,\ 0 < \eta < (1-\xi)\sqrt{3}\}$ (see Figure 3.2) we denote the standard square and standard triangle respectively. Further, let $P_p^1(T)$ be the set of all polynomials of degree $\le p$ on T and $P_p^2(S)$ the minimal set of polynomials on S consisting of all polynomials of degree $\le p$ and polynomials which are of degree $\le p$ on one side and are zero on the three others. For details see eg. [20]. The set $P_p^2(S)$ is the set of serendipity elements and in [21] is denoted by Q_p'.

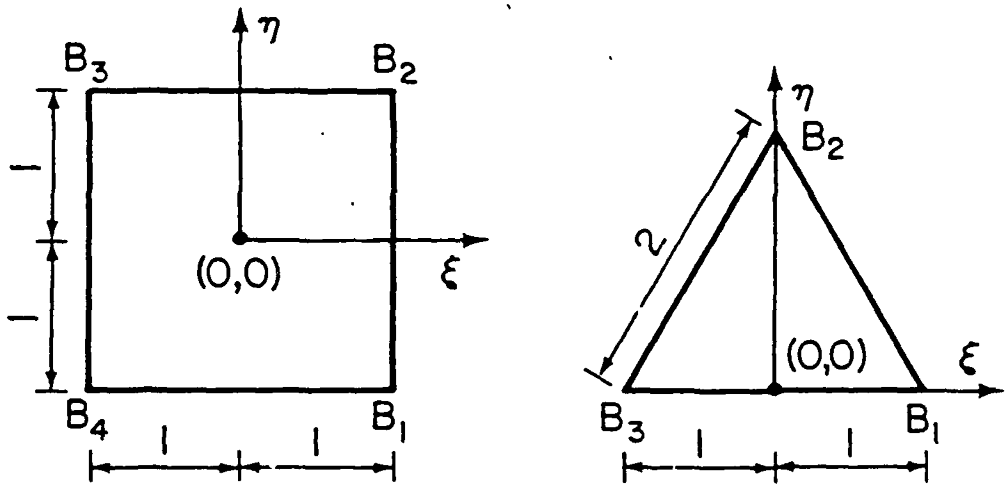

Figure 3.2. The scheme of S and T

Assume now that the mappings $F_j := (x_i^{[j]} = x_i^{[j]}(\xi,\eta),\ i = 1,2)$, $j = 1,\dots,q$ are smooth one-to-one mappings of $\bar{S}$ on $\bar{\Omega}_j$ if Ω_j is a (curved) quadrilateral and $\bar{T}$ on $\bar{\Omega}_j$ if Ω_j is a (curved) triangle. We can now speak about vertices and sides of Ω_j in the obvious sense.

We will assume the following about the partition of Ω and the mappings F_j:

i) The intersection $\bar{\Omega}_i \cap \bar{\Omega}_j$ is either empty or is the single common vertex of Ω_i and Ω_j or is the single entire side of Ω_i and Ω_j.

ii) If $\Gamma_{ij} = \bar{\Omega}_i \cap \bar{\Omega}_j$ is the common side of Ω_i and Ω_j, $P \in \Gamma_{ij}$, $P = F_i(P_i) = F_j(P_j)$, $P_i \in \overline{B_k B_{k+1}}$, $P_j \in \overline{B_\ell B_{\ell+1}}$, ($B_j$ is the vertex of S or T), then $d(P_i, B_k) = d(P_j, B_{\ell+1})$ where we denoted by $d(P_i, B_k)$ the Euclidean distance between P_i and B_k.

We also will identify the side Γ_{ij} with $I = (-1,1)$ and the map $F_i^* = F_j^* = F_{ij}^*$ of I onto Γ_{ij}. Realizing that the sides of T and S have length $= 2$ the relation between F_i, F_i^*, F_{ij}^* is obvious.

Let now $P_p(\Omega) = \{u \in H^1(\Omega) | u(F_i^{-1}(x_1,x_2)) \in P_p^1(T)$ if Ω_i is a (curved) triangle, $u(F_i^{-1}(x_1,x_2)) \in P_p^2(S)$ if Ω_i is a (curved) quadrilateral$\}$. If $u \in P_p(\Omega)$ then $u(F_i^{-1}(x_1,x_2))$ is a polynomial of degree p on every side of T, respectively S. We will identify the sides of T, respectively S, with $I = (-1,1)$ in the obvious way.

Assume now that a mapping $\mathring{\mathcal{R}}_p$ is given which maps $H^k(I)$, $k > 1/2$, onto $P_p(I)$, (i.e., the set of polynomials of degree p on I), with $(\mathring{\mathcal{R}}_p u)(\pm 1) = u(\pm 1)$ and $\mathring{\mathcal{R}}_p u = u$ for $u \in P_p(I)$. We note that for $k > 1/2$, $H^k(I) \hookrightarrow C(\bar{I})$ and hence $u(\pm 1)$ is well defined.

The p-version of the finite element method for solving problem (3.1) consists of finding $u_p \in P_p(\Omega)$ such that

(3.2a) $$B(u_p, v) = F(v) \quad \forall\, v \in P_p(\Omega) \cap H_0,$$

(3.2b) $$u_p = \mathfrak{K}_p^{[i]} g^{[i]} \quad \text{on} \quad \Gamma_i, \quad i \in \mathcal{D},$$

where

$$\mathfrak{K}_p^{[i]} g^{[i]}(x) = F_i^* \mathfrak{K}_p(g^{[i]}(F_i^{*-1}(x)))$$

and F_i^* is the mapping of I on Γ_i induced by the mapping F_i.

The following theorem gives an error estimate for the p-version.

Theorem 3.1. Let the solution u_0 of the problem (3.1) have the form

(3.3a) $$u_0 = u_1 + \sum_{j=1}^{M} \sum_{i=1}^{n_j} c_j^{[i]} u_j^{[i]}$$

(3.3b) $$u_1 \in H^k(\Omega), \quad k > 1,$$

(3.3c) $$u_j^{[i]} = r_j^{\alpha_j^{[i]}} |\log r_j|^{\gamma_j^{[i]}} \chi_j^{[i]}(r_j) \Phi_j^{[i]}(\theta_j)$$

where (r_j, θ_j) are the polar coordinates with the origin at A_j, $\alpha_j^{[i+1]} \ge \alpha_j^{[i]} > 0$, $\gamma_j^{[i]} \ge 0$, $\chi_j^{[i]}(r_j)$ is a C^∞ cut-off function and $\Phi_j^{[i]}$ is a C^∞ function in θ_j. Then

(3.4) $$\|u_0 - u_p\|_{H^1(\Omega)} \le C\Big[p^{-(k-1)} \|u_1\|_{H^k(\Omega)} + p^{-\mu} |\log p|^s \sum_{i,j} |c_j^{[i]}| + \sum_{i \in \mathcal{D}} \|g^{[i]} - \mathfrak{K}_p^{[i]} g^{[i]}\|_{H_{00}^{1/2}(\Gamma_i)}\Big]$$

where $\mu = \min_{i,j}(2\alpha_j^{[i]})$, $s = \gamma_j^{[i]}$ where $\mu = 2\alpha_j^{[i]}$, C is a constant independent of u_0 and p but dependent on k, μ, s and on the partition of Ω.

Proof. The proof of (3.4) is essentially the same as of Theorem 4.1 in [11]. $\square$

Let us mention that

$$\|g^{[i]} - \mathcal{R}_p^{[i]} g^{[i]}\|_{H_{00}^{1/2}(\Gamma_i)} \approx \|g^{[i]*} - \mathcal{R}_p g^{[i]*}\|_{H_{00}^{1/2}(I)}$$

where $g^{[i]*}(\xi) = g^{[i]}(F_i^*(\xi))$, $\xi \in I$, because by the assumption $F_i(\xi)$ is smooth.

Theorem 3.1 shows that the error of the finite element solution depends on two terms. The first two terms in (3.4) represent the error of the best approximation without the effect of imposing the Dirichlet boundary condition. In the case when $\mathcal{D} = \phi$ or $g^{[i]} = 0$ only these terms are present in (3.4). The last term shows the effect of the approximation of the Dirichlet condition in the dependence on the projection $\mathcal{R}_p$. We will call this term the <u>constraint term</u>. We defined in Section 2.4 the projections $\mathcal{P}_p^{\alpha}$ and $\mathcal{P}_p^{\circ\alpha}$ for the set of functions u which satisfy the condition $u(\pm 1) = 0$. In order to choose $\mathcal{R}_p = \mathcal{P}_p^{\alpha}$ we define

$$\mathcal{R}_p^{\alpha} u^* = \bar{u}^* + \mathcal{P}_p^{\alpha}(u^* - \bar{u}^*)$$

where $u^*(\xi) = u(F_i^*(\xi))$, $\xi \in I$, and $\bar{u}^*$ is the linear interpolant (in ξ) of u^* so that $(u^* - \bar{u}^*)(\pm 1) = 0$. The mapping $\mathcal{R}_p^{\circ\alpha}$ is defined similarly. For a simplification of the notation we will write often $\mathcal{P}_p^{\alpha}$ and $\mathcal{P}_p^{\circ\alpha}$ instead of $\mathcal{R}_p^{\alpha}$ and $\mathcal{R}_p^{\circ\alpha}$.

We can now directly combine Theorems 2.9, 2.10, 2.12, and Theorem 3.1 to get the error estimate for the p-version of the finite element method when using $\mathcal{R}_p^{\alpha} \approx \mathcal{P}_p^{\alpha}$ or $\mathcal{R}_p^{\circ\alpha} \approx \mathcal{P}_p^{\circ\alpha}$. For example, let $1 \ge \alpha > 1/2$, $s > \alpha$, $\mathcal{R}_p^{\circ\alpha} \approx \mathcal{P}_p^{\circ\alpha}$. Then the constraint term has the estimate

(3.5) $$\|g^{[i]} - \mathcal{R}_p^{\circ\alpha} g^{[i]}\|_{H_{00}^{1/2}(\Gamma_i)} \le C(\varepsilon) p^{-(s-(1/2)-\varepsilon)} \|g^{[i]*} - \bar{g}^{[i]*}\|_{\mathring{W}^s(\alpha - s, \alpha)}$$

where $g^{[i]*}$ and $\bar{g}^{[i]*}$ were defined above. We assume that $s \neq [s]+1/2$ and $\alpha+s \neq 1+2k$, $k = 0,\dots,[s]-1$. We obviously "lost" a power p^{ε} in (3.5) in comparison with the optimal estimate.

We have assumed that $\alpha > 1/2$. If $\alpha < 1/2$ then we have to apply Theorem 2.12 and get on the right hand side of (3.5) the term

$$p^{-(s-(1/2)-\varepsilon)+((1/2)-\alpha)}\|g^{[i]*}-\bar{g}^{[i]*}\|_{\hat{W}((1/2)+\varepsilon-s,(1/2)+\varepsilon)}.$$

3.3. <u>Numerical experiments</u>.

Let us consider the case of an L-shaped domain shown in Figure 3.1. We will partition this domain into three squares as indicated also in Figure 3.3.

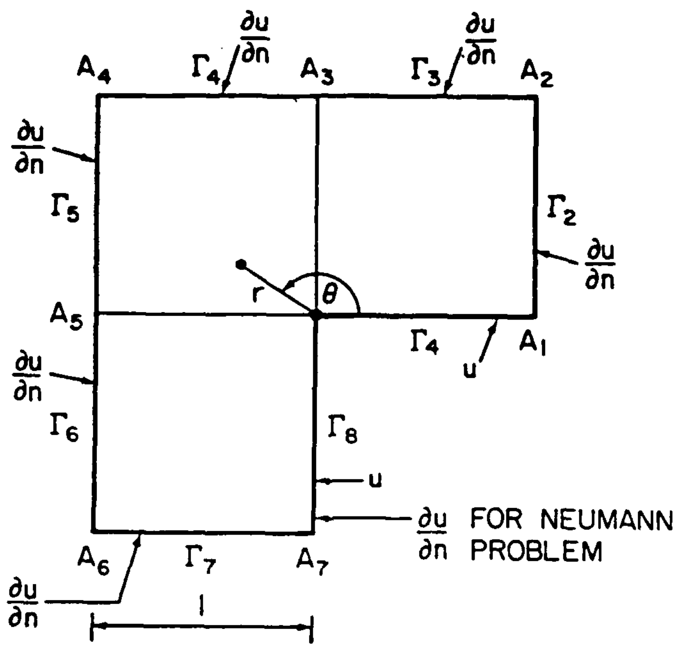

Figure 3.3. The L-shaped domain.

Let us consider the model problem

(3.6a) $$\Delta u = 0,$$

(3.6b) $$u = g \text{ on } \Gamma_1 \text{ and } \Gamma_8 \quad (\text{i.e., } 1,8 \in \mathcal{D}),$$

(3.6c) $$\frac{\partial u}{\partial n} = h \text{ on } \Gamma_i, \quad 2 \le i \le 7.$$

We will assume that g and h are such that the exact solution u_0 is

$$u_0 = r^{\lambda} \sin \lambda\theta \tag{3.7}$$

where (r,θ) are the polar coordinates with the origin at the vertex A_1 (see Figure 3.1). Note that $u_0 = 0$ on Γ_1. This problem will be called the <u>Dirichlet</u> problem. In addition we will consider the <u>Neumann</u> problem when $\mathcal{D} = \{1\}$. The problem now is formulated in the form (3.1) with

$$B(u,v) = \int_{\Omega} \left[\sum_{i=1}^{2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \right] dx$$

$$F(v) = \int_{\Gamma_i} hvdx \quad i \in \mathcal{M} - \mathcal{D}.$$

The constraint term is not present in the Neumann problem. It is present on Γ_8 in the case of the Dirichlet problem

We will now use the projection $\mathcal{R}_p^{\circ\alpha}$ which is advantageous in practice (although in some cases some assumptions in our theorems are violated).

The Neumann problem has no constraint term and hence the error for the Dirichlet problem has to be larger. The difference indicates the influence of the constraint term (and of $\mathcal{R}_p^{\circ\alpha}$) on the accuracy of the solution. In Tables 3.1 - 3.5 we show the relative error $|u_0 - u_p|_{H^1(\Omega)} / |u_0|_{H^1(\Omega)}$ for various α and λ.

We see that in the case $\lambda = 1.6$ and $\lambda = 0.6$ the ratio of the error of the Neumann problem and the Dirichlet problem is a reasonable one for all α and nearly independent of p. In [7] we have proven that in this case, when $g^{[8]} \in H^1(\Gamma_8)$, the rate for the Neumann and Dirichlet problem is the optimal one when $\mathcal{P}_p^{\circ 1}$ (or $\mathcal{R}_p^{\circ 1}$) was used. We cannot theoretically explain why the

error decreases with $\alpha \to 0$ for $\alpha < 1/2$. Very likely it relates to the used range $p \le 8$ and analogous effects we have seen in Figure 2.2.

In the case $\lambda = 0.15$ when $g^{[8]} \notin H^1(\Gamma_8)$ we see the divergence when $\mathcal{P}_p^{\circ 1}$ is used, as expected, but once more we cannot explain why $\mathcal{P}_p^{\circ 1/4}$ gives best results. We see that $\mathcal{P}_p^{\circ 0}$ performs slightly worse than $\mathcal{P}_p^{\circ 1/4}$ (but still better than $\mathcal{P}_p^{\circ 1/2}$).

In the case $\lambda = 0.05$, the use of $\mathcal{P}_p^{\circ 0}$ leads to the divergence.

A natural question arises. What α should be used in practice? The experiment suggests the choices $\alpha \le 1/2$. $\alpha = 1/2$ should be preferred for theoretical reasons but $\alpha = 0$ usually gives the best results in our experiments (these and others) and never gives a bad result. We note that for $\lambda = 0.05$, the p-version gives unacceptable results even for the Neumann problem and hence the h-p version has to be used.

If the h-p version (or p-version with strongly refined meshes) is used, then the difference between various projection operators is not too important. We will illustrate one such choice of operator when Dirichlet conditions are imposed on the entire boundary. The solution is $u = r^{1/3} \sin \theta/3$ (case A) and $u = r^{2/3} \cos 2\theta/3$ (case B). The projection $\mathcal{P}_p^{\circ 1}$ is used. Figure 3.4 (case A) and 3.5 (case B) show the relative error for different p and the meshes which are strongly refined. They have n layers and for $n = 1$, the mesh shown in Figure 3.2. Theoretically (see [9], [10]), the h-p version converges exponentially as $e^{-\gamma\sqrt[3]{N}}$ and hence Figures 3.4 and 3.5 are plotted in the $\log \times N^{1/3}$ scale. We see also in case A that the h-p version with $n = p$ converges in fact as $e^{-\gamma\sqrt[3]{N}}$.

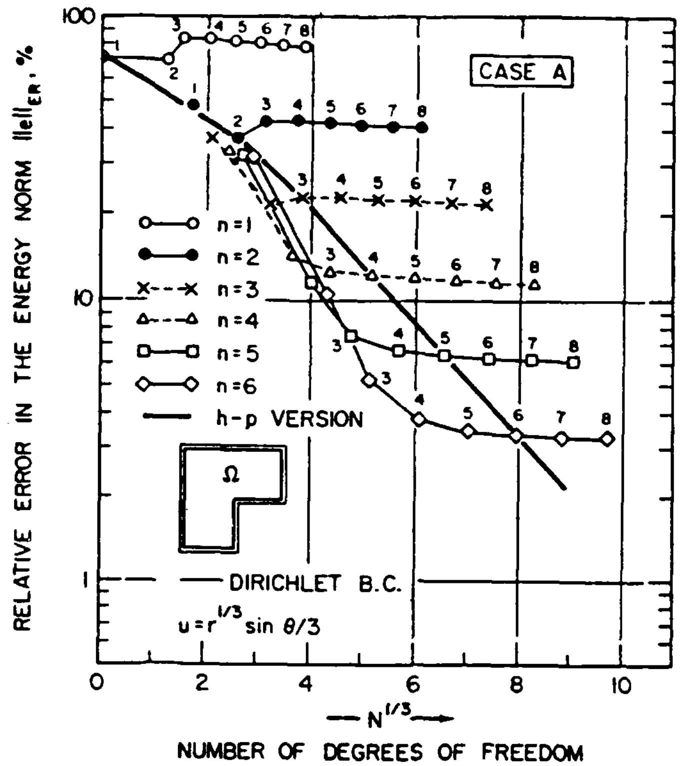

Figure 3.4. The error of the p and h-p versions for $u = r^{1/3} \sin \theta/3$.

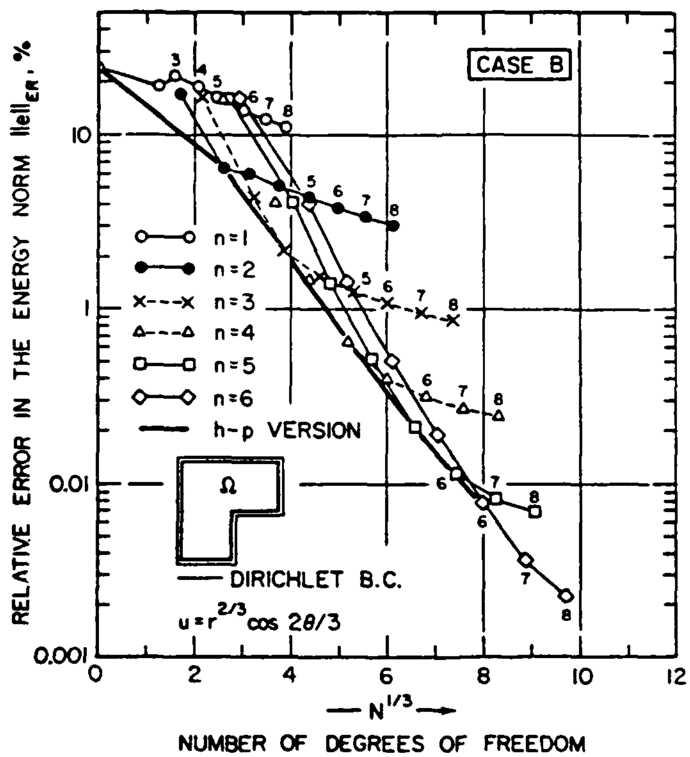

Figure 3.5. The error of the p and h-p version for $u = r^{2/3} \cos 2\theta/3$.

Table 3.1. Relative error for $\lambda = 1.6$

p	$\mathscr{P}_p^{\circ 0}$	$\mathscr{P}_p^{\circ 1/4}$	$\mathscr{P}_p^{\circ 1/2}$	$\mathscr{P}_p^{\circ 3/4}$	$\mathscr{P}_p^{\circ 1}$	NBC
1	2.6001 E-1	2.6001 E-2	2.6001 E-1	2.6001 E-1	2.6001 E-1	2.5761 E-1
2	3.6361 E-2	3.6297 E-2	3.6232 E-2	3.6171 E-2	3.6127 E-2	3.1326 E-2
3	3.0482 E-2	3.0422 E-2	3.0367 E-2	3.0331 E-2	3.0340 E-2	2.3647 E-2
4	1.5320 E-2	1.5361 E-2	1.5426 E-2	1.5529 E-2	1.5699 E-2	1.2535 E-2
5	8.2123 E-3	8.2512 E-3	8.3240 E-3	8.4285 E-3	8.5922 E-3	7.1002 E-3
6	4.8521 E-3	4.8886 E-3	4.9435 E-3	5.0260 E-3	5.1533 E-3	4.2937 E-3
7	3.0914 E-3	3.1174 E-3	3.1571 E-3	3.2182 E-3	3.3137 E-3	2.7834 E-3
8	2.0821 E-3	2.1029 E-3	2.1338 E-3	2.1809 E-3	2.2546 E-3	1.8983 E-3

Table 3.2. Relative error for $\lambda = 0.6$

p	$\mathscr{P}_p^{\circ 0}$	$\mathscr{P}_p^{\circ 1/4}$	$\mathscr{P}_p^{\circ 1/2}$	$\mathscr{P}_p^{\circ 3/4}$	$\mathscr{P}_p^{\circ 1}$	NBC
1	3.0707 E-1	3.0707 E-1	3.0707 E-1	3.0707 E-1	3.0707 E-1	2.6397 E-1
2	2.1588 E-1	2.1671 E-1	2.1795 E-1	2.2008 E-1	2.2436 E-1	1.7145 E-1
3	2.1377 E-1	2.1513 E-1	2.1720 E-1	2.2267 E-1	2.2781 E-1	1.5878 E-1
4	1.5851 E-1	1.5991 E-1	1.6203 E-1	1.6563 E-1	1.7289 E-1	1.2927 E-1
5	1.2400 E-1	1.2528 E-1	1.2726 E-1	1.3065 E-1	1.3763 E-1	1.0516 E-1
6	1.0116 E-1	1.0233 E-1	1.0414 E-1	1.0725 E-1	1.1378 E-1	8.7797 E-2
7	8.5071 E-2	8.6129 E-2	8.7780 E-2	9.0682 E-2	9.6911 E-2	7.4923 E-2
8	7.3162 E-2	7.4134 E-2	7.5652 E-2	7.8361 E-2	8.4305 E-2	6.5076 E-2

Table 3.3. Relative error for $\lambda = 0.3333$

p	$\wp_p^{\circ 0}$	$\wp_p^{\circ 1/4}$	$\wp_p^{\circ 1/2}$	$\wp_p^{\circ 3/4}$	$\wp_p^{\circ 1}$	NBC
1	7.1888 E-1	7.1888 E-1	7.1888 E-1	7.1888 E-1	7.1888 E-1	5.7124 E-1
2	5.8300 E-1	5.8538 E-1	5.9181 E-1	6.1187 E-1	7.0125 E-1	4.7273 E-1
3	5.8462 E-1	5.9263 E-1	6.1026 E-1	6.5729 E-1	8.3462 E-1	4.5779 E-1
4	4.9954 E-1	5.1145 E-1	5.3729 E-1	6.0323 E-1	8.3153 E-1	4.1310 E-1
5	4.4324 E-1	4.5536 E-1	4.8347 E-1	5.5783 E-1	8.1688 E-1	3.6981 E-1
6	3.9661 E-1	4.0957 E-1	4.4009 E-1	5.2086 E-1	8.0127 E-1	3.3549 E-1
7	3.6438 E-1	3.7673 E-1	4.0747 E-1	4.9119 E-1	7.8886 E-1	3.3079 E-1
8	3.3501 E-1	3.4756 E-1	3.7924 E-1	4.6603 E-1	7.8025 E-1	2.8517 E-1

Table 3.4. Relative error for $\lambda = 0.15$

p	$\wp_p^{\circ 0}$	$\wp_p^{\circ 1/4}$	$\wp_p^{\circ 1/2}$	$\wp_p^{\circ 3/4}$	$\wp_p^{\circ 1}$	NBC
1	1.0123 E-0	1.0123 E-0	1.0123 E-0	1.0123 E-0	1.0123 E-0	8.5288 E-1
2	9.1644 E-1	9.1240 E-1	9.0754 E-1	9.0775 E-1	1.0672 E-0	7.8644 E-1
3	9.0381 E-1	9.0073 E-1	9.0034 E-1	9.2422 E-1	1.3136 E-0	7.7035 E-1
4	8.4599 E-1	8.4380 E-1	8.4864 E-1	8.9877 E-1	1.4699 E-0	7.3960 E-1
5	8.1341 E-1	8.1065 E-1	8.1748 E-1	8.8487 E-1	1.5893 E-0	7.0663 E-1
6	7.7966 E-1	7.7722 E-1	7.8714 E-1	8.7191 E-1	1.6836 E-0	6.7779 E-1
7	7.5917 E-1	7.5626 E-1	7.6744 E-1	8.6455 E-1	1.7626 E-0	6.5296 E-1
8	7.3550 E-1	7.3276 E-1	7.4608 E-1	8.5618 E-1	1.8308 E-0	6.3139 E-1

Table 3.5. Relative error for $\lambda = 0.05$

p	$\vartheta_p^{\circ 0}$	NBC
1	1.0306 E-0	9.6485 E-1
2	1.0851 E-0	9.4059 E-1
3	1.2706 E-0	9.3285 E-1
4	1.4422 E-0	9.2138 E-1
5	1.5993 E-0	9.0854 E-1
6	1.7407 E-0	8.9661 E-1
7	1.8707 E-0	8.8591 E-1
8	1.9920 E-0	8.7630 E-1

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